Concurrence of assistance and Mermin inequality on three-qubit pure states

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We study a relation between the concurrence of assistance and the Mermin inequality on three-qubit pure states. We find that if a given three-qubit pure state has the minimal concurrence of assistance greater than 1/2 then the state violates some Mermin inequality.

Bell-inequality violation in quantum mechanics tells us that quantum correlations are quite different from classical correlations. In the case of two-qubit states, the Clauser-Horne-Shimony-Holt (CHSH) inequality [1] is a well-known Bell inequality, and has an important property that any two-qubit pure state violates the CHSH inequality if and only if it is entangled. In particular, there exists an explicit relation between the degree of the CHSH-inequality violation and the amount of entanglement for two-qubit pure states [2]. This shows that entanglement of pure states in the two-qubit system can be certainly detected by employing the Bell inequality, and the Bell-inequality violation can be exactly determined according to the amount of entanglement for two-qubit pure states. On this account, there have been a lot of research works to attempt to generalize the explicit relation into the multiqubit pure states [3, 4, 5, 6, 7].

We here consider the Mermin inequality [8] for three-qubit pure states, which is a natural generalization of the CHSH inequality: Let $\mathcal{B}_M$ be the operator defined as

$$\mathcal{B}_M = \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma} - \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} \\ -\vec{b}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} - \vec{b}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma}, \quad (1)$$

where $\vec{a}_j$ and $\vec{b}_j$ are unit vectors in $\mathbb{R}^3$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of the Pauli matrices. Then for a given three-qubit pure state $|\psi\rangle$, the Mermin inequality is

$$|\langle\psi|\mathcal{B}_M|\psi\rangle| \leq 2. \quad (2)$$

For the generalized Greenberger-Horne-Zeilinger (GHZ) states,

$$|\psi_{\text{GHZ}}\rangle = \cos\phi|000\rangle + \sin\phi|111\rangle, \quad (3)$$

it was numerically shown in Ref. [3] that the state $|\psi_{\text{GHZ}}\rangle$ violates a Mermin inequality if and only if $\sin 2\phi > 1/2$. This result implies that there exists a relation between the Mermin-inequality violation and the amount of entanglement for three-qubit pure states, since $\sin 2\phi$ may represent the degree of entanglement in the state $|\psi_{\text{GHZ}}\rangle$. Then one could naturally ask whether the same result can be obtained for any three-qubit pure state.

In order to answer this question, the proper quantity such as the value $\sin 2\phi$ in the generalized GHZ states should be defined for general three-qubit pure states, and it should be investigated whether the Mermin inequality is violated, whenever the quantity is greater than some constant. In this paper, we consider the *concurrence of assistance* (CoA) [9] as such a quantity, and examine a relation between the CoA and the Mermin-inequality violation for several classes of three-qubit pure states including the generalized GHZ states, the states in the W class, and some coherent superpositions of well-known three-qubit pure states.

As a consequence, we analytically show that if a three-qubit pure state in those classes has the minimal CoA greater than 1/2 then the state violates a Mermin inequality, and furthermore find that our result can be generalized into all three-qubit pure states by exploiting the numerical work in Ref. [5].

We first take account of two simple but important measures of entanglement, the concurrence [10] and the CoA. The concurrence, $\mathcal{C}$, is defined as follows: For a pure state $|\phi\rangle_{12}$ in $2 \otimes d$ quantum systems ($d \geq 2$), it is defined as $\mathcal{C}(|\phi\rangle_{12}\langle\phi|) = \sqrt{2(1 - \text{tr}\rho_1^2)} = 2\sqrt{\det\rho_1}$, where $\rho_1 = \text{tr}_2|\phi\rangle_{12}\langle\phi|$. For any mixed state ρ_{12}, it is defined as

$$\mathcal{C}(\rho_{12}) = \min \sum_k p_k \mathcal{C}(|\phi_k\rangle_{12}\langle\phi_k|), \quad (4)$$

where the minimum is taken over its all possible decompositions, $\rho_{12} = \sum_k p_k |\phi_k\rangle_{12}\langle\phi_k|$. The CoA, $\mathcal{C}^a$, is also defined in the similar way: For a pure state $|\phi\rangle_{12}$, $\mathcal{C}^a(|\phi\rangle_{12}\langle\phi|) \equiv \mathcal{C}(|\phi\rangle_{12}\langle\phi|)$. For a mixed state ρ_{12}, it is defined as

$$\mathcal{C}^a(\rho_{12}) = \max \sum_k p_k \mathcal{C}(|\phi_k\rangle_{12}\langle\phi_k|), \quad (5)$$

where the maximum is taken over all possible decompositions of ρ_{12}.

We remark that the CoA is an entanglement monotone on three-qubit pure states [11]. Thus, even though the definitions of the two entanglement measures are quite similar, the CoA can be thought of as a measure of entanglement on tripartite pure states, while the concurrence is a good measure of bipartite entanglement. Our aim in

this paper is to define an appropriate measure of entanglement for three-qubit pure states, and to investigate how the entanglement measure is related to the Mermin-inequality violation. Hence, the CoA may be one of good candidates for such a tripartite entanglement measure.

Furthermore, it was known in Ref. [12] that, for any three-qubit pure state $|\psi\rangle_{123}$, there exists the so-called monogamy equality in terms of the concurrence and the CoA as follows:

$$\mathcal{C}_{1(23)}^2 = \mathcal{C}_{12}^2 + (\mathcal{C}_{13}^a)^2, \qquad (6)$$

where $\mathcal{C}_{1(23)} = \mathcal{C}(|\psi\rangle_{1(23)}\langle\psi|)$, $\mathcal{C}_{12} = \mathcal{C}(\mathrm{tr}_3|\psi\rangle_{123}\langle\psi|)$, and $\mathcal{C}_{13}^a = \mathcal{C}^a(\mathrm{tr}_2|\psi\rangle_{123}\langle\psi|)$. Thus, for distinct i and j in $\{1, 2, 3\}$, we clearly have the equalities $\mathcal{C}_{ij}^a = \sqrt{\tau + \mathcal{C}_{ij}^2}$, where τ is called the three-tangle [13, 14], defined as

$$\tau = \mathcal{C}_{1(23)}^2 - \mathcal{C}_{12}^2 - \mathcal{C}_{13}^2, \qquad (7)$$

and is known as an entanglement measure to distinguish the GHZ class from the W class [14]. Here, the GHZ class and the W class are the sets of all pure states with genuine three-qubit entanglement equivalent to the GHZ state [15],

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right), \qquad (8)$$

under stochastic local operations and classical communication (SLOCC), and equivalent to the W state,

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right), \qquad (9)$$

under SLOCC, respectively.

Now, we consider the Schmidt decomposition of three-qubit pure states as follows [16]:

$$\begin{aligned} |\psi\rangle_{123} &= \lambda_0|000\rangle_{123} + \lambda_1 e^{\iota\theta}|100\rangle_{123} + \lambda_2|101\rangle_{123} \\ &\quad + \lambda_3|110\rangle_{123} + \lambda_4|111\rangle_{123}, \end{aligned} \qquad (10)$$

where $\iota = \sqrt{-1}$, $0 \leq \theta \leq \pi$, $\lambda_j \geq 0$, and $\sum_j \lambda_j^2 = 1$. Thus, in order to calculate the CoAs for three-qubit pure states, it suffices to consider the states in Eq. (10). By somewhat tedious but straightforward calculations, we obtain the following results on the CoAs $\mathcal{C}_{ij}^a$ for $|\psi\rangle_{123}$:

$$\begin{aligned} \mathcal{C}_{12}^a &= 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2}, \\ \mathcal{C}_{23}^a &= 2\sqrt{\lambda_0^2\lambda_4^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4\cos\theta}, \\ \mathcal{C}_{31}^a &= 2\lambda_0\sqrt{\lambda_2^2 + \lambda_4^2}. \end{aligned} \qquad (11)$$

Let $\mathcal{C}_{\min}^a = \min\{\mathcal{C}_{12}^a, \mathcal{C}_{23}^a, \mathcal{C}_{31}^a\}$. Then $\mathcal{C}_{\min}^a$ is called the minimal CoA, which is the very entanglement measure relevant to our purpose. Our claim is that, for a given three-qubit pure state, if its minimal CoA is greater than $1/2$ then there exists a Mermin inequality which the state violates.

For the generalized GHZ states $|\psi_{\mathrm{GHZ}}\rangle$ in Eq. (3), it is easy to calculate that $\mathcal{C}_{\min}^a = \sin 2\phi$ by Eqs. (11). Take $\vec{a}_j = (1, 0, 0)$ and $\vec{b}_j = (0, 1, 0)$ for all $j = 1, 2, 3$. Then the Mermin inequality in Eq. (2) becomes $4\sin 2\phi \leq 2$. Hence, we can clearly obtain that if the generalized GHZ state has the minimal CoA greater than $1/2$ then the state violates a Mermin inequality. In other words, our claim is true for the generalized GHZ states.

We now take the W class into account. It is known in [14, 17] that any state $|\psi_{\mathrm{W}}\rangle$ in the W class can be written as

$$|\psi_{\mathrm{W}}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle, \qquad (12)$$

which has the simpler Schmidt decomposition than the general one in Eq. (10), since its three-tangle is zero. In this case, we take $\vec{a}_j = (0, 0, 1)$ for all $j = 1, 2, 3$, $\vec{b}_1 = (-1, 0, 0)$, and $\vec{b}_2 = \vec{b}_3 = (1, 0, 0)$. Then the Mermin inequality in Eq. (2) becomes

$$\lambda_0^2 - \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_0\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_0 \leq 2. \qquad (13)$$

By Eqs. (11), the minimal CoA, $\mathcal{C}_{\min}^a$, for the W class can be readily calculated as

$$\mathcal{C}_{\min}^a = 2\min\{\lambda_0\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_0\}. \qquad (14)$$

Since $\lambda_i^2 + \lambda_j^2 \geq 2\lambda_i\lambda_j$ and $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, we can obtain the following inequalities:

$$\begin{aligned} 2(1 - \lambda_1^2) &= 2(\lambda_0^2 + \lambda_2^2 + \lambda_3^2) \\ &\geq 2\lambda_0\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_0 \\ &\geq 3\mathcal{C}_{\min}^a. \end{aligned} \qquad (15)$$

Thus, if $\mathcal{C}_{\min}^a > 1/2$ then $\lambda_1^2 < 1/4$, and hence the left-hand side of the inequality in (13) is greater than 2, that is, its Mermin inequality is violated. Therefore, our claim is also true for the W class.

We now consider three coherent superpositions of well-known states. First, let us see a coherent superposition of the generalized GHZ state and a separable state $|101\rangle$,

$$|\psi_{\mathrm{GHZ}} : \mathrm{S}\rangle = \sqrt{1-p}|\psi_{\mathrm{GHZ}}\rangle + \sqrt{p}|101\rangle, \qquad (16)$$

where $0 < p < 1$. Taking account of the Mermin inequality used in the case of the generalized GHZ states, we can show that the Mermin inequality is violated if and only if $4(1-p)\sin 2\phi > 2$, and that the minimal CoA for $|\psi_{\mathrm{GHZ}} : \mathrm{S}\rangle$ equals $(1-p)\sin 2\phi$ by Eqs. (11). Hence, it is clear that if the minimal CoA for the state $|\psi_{\mathrm{GHZ}} : \mathrm{S}\rangle$ is more than $1/2$ then it violates the same Mermin inequality as the inequality for the generalized GHZ states, and vice versa.

The second coherent superposition which we deal with is a superposition of the W state and a separable state $|000\rangle$,

$$|\mathrm{W} : \mathrm{S}\rangle = \sqrt{1-p}|\mathrm{W}\rangle + \sqrt{p}|000\rangle, \qquad (17)$$
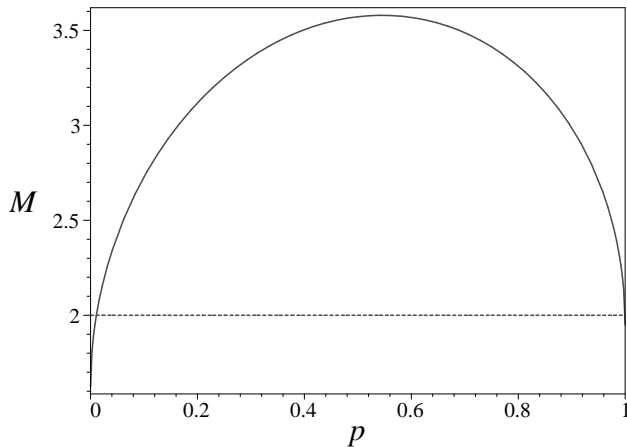
FIG. 1: The left-hand side $M(p)$ of the inequality in (19) is more than two for $0.011 \leq p \leq 0.999$.

where $0 < p < 1$. We now take $\vec{a}_j = (0, 0, -1)$ and $\vec{b}_j = (1, 0, 0)$ for all $j = 1, 2, 3$. Then the Mermin inequality becomes $3 - 4p \leq 2$, that is, $p \geq 1/4$. It can be obtained from simple calculations [18] that its minimal CoA is $2(1 - p)/3$. Thus, the minimal CoA for the state $|W : S\rangle$ is greater than $1/2$ if and only if the Mermin inequality with respect to $\vec{a}_j = (0, 0, -1)$ and $\vec{b}_j = (1, 0, 0)$ is violated.

Let us now deal with the coherent superposition of the GHZ state and the W state,

$$|\text{GHZ} : \text{W}\rangle = \sqrt{1 - p}|\text{GHZ}\rangle + \sqrt{p}|\text{W}\rangle, \qquad (18)$$

where $0 < p < 1$. Then it follows from direct computations that for all $0 < p < 1$ the states $|\text{GHZ} : \text{W}\rangle$ have the minimal CoA more than $1/2$. Thus, it suffices to show that the state violates some Mermin inequality for each $0 < p < 1$. We here use three Mermin inequalities to show the violation, according to the value of the parameter p. We first consider the Mermin inequality with respect to $\vec{a}_j = (1/2, 0, -\sqrt{3}/2)$ and $\vec{b}_j = (0, 1, 0)$ for all $j = 1, 2, 3$. Then the Mermin inequality for the states $|\text{GHZ} : \text{W}\rangle$ becomes

$$\frac{1}{8}\left(3\sqrt{2}(5 + \sqrt{3})\sqrt{p(1 - p)} + 13(1 - p) + 9\sqrt{3}p\right) \leq 2. \qquad (19)$$

Let $M(p)$ be the left-hand side of the inequality in (19). Then it can be shown that $M(p)$ is greater than two if $0.011 \leq p \leq 0.999$, and hence the Mermin inequality is violated in this case, which is depicted in FIG. 1.

Furthermore, it can be readily shown that the states $|\text{GHZ} : \text{W}\rangle$ for $0 < p < 1/2$ and $1 > p > (9 + \sqrt{21})/15 \simeq 0.9055$ violate the Mermin inequalities taken in the cases of the states $|\psi_{\text{GHZ}}\rangle$ and the states $|\text{W} : \text{S}\rangle$, respectively. Therefore, our claim also holds for the states $|\text{GHZ} : \text{W}\rangle$.

In addition to our analytical results, our claim can be generalized into any three-qubit pure states, by exploiting the numerical work of Emary and Beenakker in Ref. [5]. In their work, an entanglement measure σ was defined as

$$\sigma \equiv \min\left(\frac{\mathcal{C}_{X(YZ)}^2 + \mathcal{C}_{Y(XZ)}^2}{2} - \mathcal{C}_{XY}^2\right), \qquad (20)$$

where the minimization is over the permutations X, Y, Z in $\{1, 2, 3\}$. Then we can obtain the simple relation among the three-tangle, the minimal CoA, and this measure of entanglement σ as follows:

$$0 \leq \tau \leq (\mathcal{C}_{\min}^a)^2 \leq \sigma \leq 1. \qquad (21)$$

Since their numerical result shows the following inequality

$$\sigma \leq \frac{|\langle \psi | \mathcal{B}_M | \psi \rangle|^2}{16} \qquad (22)$$

for numbers of three-qubit pure states $|\psi\rangle$, this implies that our claim is numerically true for general three-qubit pure states.

In summary, we have studied a relation between the CoA and the Mermin-inequality violation for several classes of three-qubit pure states, and have obtained an analytical result that if a three-qubit pure state in those classes has the minimal CoA greater than $1/2$ then the state violates some Mermin inequality. Furthermore, we have also found that our result numerically holds for any three-qubit pure states.

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